Trig Substitutions

Notes

For integrals containing this expression	Consider this substitution
$a^2 - b^2x^2$	$x = \frac{a}{b}\sin(\theta)$
$a^2 + b^2x^2$	$x = \frac{a}{b}\tan(\theta)$
$b^2x^2 - a^2$	$x = \frac{a}{b}\sec(\theta)$

Example Problems

#1 $\displaystyle\int \frac{\sqrt{25x^2+49}}{9x^4}\,dx$

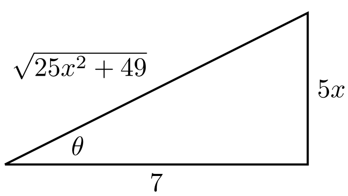
#2 $\displaystyle\int \sqrt{9-x^2}\,dx$

Example Problem #1 - Fast Work

$$\int \frac{\sqrt{25x^2+49}}{9x^4}\,dx$$

let $x = \frac{7}{5}\tan(\theta)$

then $dx = \frac{7}{5}\sec^2(\theta)\,d\theta$



$$\int \frac{\sqrt{25x^2+49}}{9x^4}\,dx$$

$$\int \frac{\sqrt{25\left(\frac{7}{5}\tan(\theta)\right)^2+49}}{9\left(\frac{7}{5}\tan(\theta)\right)^4}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

$$\int \frac{\sqrt{25\left(\frac{49}{25}\right)\tan^2(\theta)+49}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

$$\int \frac{\sqrt{49\tan^2(\theta)+49}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

$$\int \frac{\sqrt{49(\tan^2(\theta)+1)}}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; \frac{7}{5}\sec^2(\theta)\; d\theta$$

$$\int \frac{\sqrt{49\sec^2(\theta)}}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; \frac{7}{5}\sec^2(\theta)\; d\theta$$

$$\int \frac{7|\sec(\theta)|}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; \frac{7}{5}\sec^2(\theta)\; d\theta$$

Assume $\theta \in Q1$

$$\int \frac{7\sec(\theta)}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; \frac{7}{5}\sec^2(\theta)\; d\theta$$

$$\int \frac{7\sec^3(\theta)}{9\left(\frac{7}{5}\right)^3 \tan^4(\theta)} \; d\theta$$

$$\int \left(7\sec^3(\theta)\right) \div \left(9\left(\frac{7}{5}\right)^3 \tan^4(\theta)\right) \; d\theta$$

$$\int \frac{7}{\cos^3(\theta)} \div \frac{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)}{\cos^4(\theta)} \; d\theta$$

$$\int \frac{7}{\cos^3(\theta)} * \frac{\cos^4(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

$$\int \frac{7\cos(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

let $u = \sin(\theta)$

then $du = \cos(\theta)\; d\theta$

$$\int \frac{7\cos(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

$$\int \frac{7}{9\left(\frac{7}{5}\right)^3 u^4} \; du$$

$$\int \frac{7}{9\left(\frac{7}{5}\right)^3} \; u^{-4}\; du$$

$$\frac{7}{9\left(\frac{7}{5}\right)^3} \; \frac{1}{-3} u^{-3} + C$$

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}(\sin(\theta))^{-3} + C$$

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}\left(\frac{5x}{\sqrt{25x^2+49}}\right)^{-3} + C$$

$$\frac{7*5^3}{9(7)^3}\ \frac{1}{-3}\left(\frac{\sqrt{25x^2+49}}{5x}\right)^{3} + C$$

$$\frac{7*5^3}{9(7)^3}\ \frac{1}{-3}(\frac{(25x^2+49)^{3/2}}{5^3x^3}) + C$$

$$\frac{1}{9(7)^2}\ \frac{1}{-3}(\frac{(25x^2+49)^{3/2}}{x^3}) + C$$

$$-\frac{(25x^2+49)^{3/2}}{1323x^3} + C$$

Example Problem #1 - Slow Work

$$\int \frac{\sqrt{25x^2+49}}{9x^4}\ dx$$

This problem has a positive squared variable and a positive constant inside a square root, which indicates we should consider a tangent substution. Our a will be $\sqrt{49} = 7$. Because the x^2 has a coefficient of 25, our b will be $\sqrt{25} = 5$. Therefore we will use the substitution...

let $x = \frac{7}{5}\tan(\theta)$

then $dx = \frac{7}{5}\sec^2(\theta)\ d\theta$

I recommend drawing the trig subsitution's triangle as soon as we choose the trig subsitution. We begin by taking our subsitution and solving for the trig function.

$$x = \frac{7}{5}\tan(\theta)$$

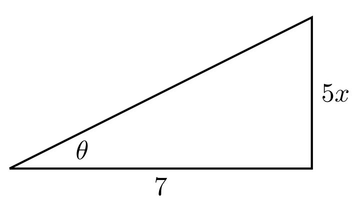
$$5x = 7\tan(\theta)$$

$$\frac{5x}{7} = \tan(\theta)$$

Remember the definition of tangent

$$\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}} = \frac{5x}{7}$$

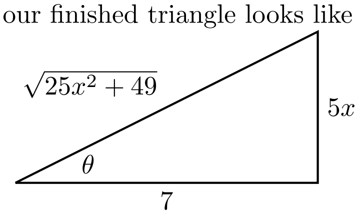
So now we draw a triangle with opposite side length $5x$ and adjacent side length 7.



The trig substution should always give us 2 sides of the triangle, and we find the third side using the Pythagorean Formula.

$$a^2 + b^2 = c^2$$
$$7^2 + (5x)^2 = c^2$$
$$49 + 25x^2 = c^2$$
$$\sqrt{49 + 25x^2} = c$$

So our finished triangle looks like this:



We will need this triangle when undoing the trig substution after we integrate. Now we actually perform the substitution.

$$\int \frac{\sqrt{25x^2+49}}{9x^4}\,dx$$

$$\int \frac{\sqrt{25\left(\frac{7}{5}\tan(\theta)\right)^2+49}}{9\left(\frac{7}{5}\tan(\theta)\right)^4}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Use the exponents.

$$\int \frac{\sqrt{25\left(\frac{49}{25}\right)\tan^2(\theta)+49}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Cancel out 25's.

$$\int \frac{\sqrt{49\tan^2(\theta)+49}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Factor out 49.

$$\int \frac{\sqrt{49(\tan^2(\theta)+1)}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Use Pythagorean identity

$$\int \frac{\sqrt{49\sec^2(\theta)}}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Take square root

$$\int \frac{7|\sec(\theta)|}{9\left(\frac{7}{5}\right)^4\tan^4(\theta)}\,\frac{7}{5}\sec^2(\theta)\,d\theta$$

Technically, the square root of square of an expression is the absolute value of that expression. But for indefinite integrals, it's generally acceptable to just say

Assume $\theta \in$ Q1

And because all trig functions are positive in Quadrant 1, we remove the absolute value bars.

$$\int \frac{7\sec(\theta)}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; \frac{7}{5}\sec^2(\theta)\; d\theta$$

Multiply fractions together

$$\int \frac{7\left(\frac{7}{5}\right)\sec^3(\theta)}{9\left(\frac{7}{5}\right)^4 \tan^4(\theta)} \; d\theta$$

Cancel a 7/5 fraction

$$\int \frac{7\sec^3(\theta)}{9\left(\frac{7}{5}\right)^3 \tan^4(\theta)} \; d\theta$$

Rewrite fraction as division

$$\int \left(7\sec^3(\theta)\right) \div \left(9\left(\frac{7}{5}\right)^3 \tan^4(\theta)\right) \; d\theta$$

Rewrite secant and tangent in terms of sines and cosines

$$\int \frac{7}{\cos^3(\theta)} \div \frac{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)}{\cos^4(\theta)} \; d\theta$$

When dividing fractions, flip the second and multiply

$$\int \frac{7}{\cos^3(\theta)} * \frac{\cos^4(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

Cancel some cosines

$$\int \frac{7\cos(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

Now that we have mostly sines with a single cosine on top, we can use a substitution.

let $u = \sin(\theta)$
then $du = \cos(\theta)\; d\theta$

$$\int \frac{7\cos(\theta)}{9\left(\frac{7}{5}\right)^3 \sin^4(\theta)} \; d\theta$$

$$\int \frac{7}{9\left(\frac{7}{5}\right)^3 u^4} \; du$$

Rewrite positive exponent on bottom of fraction as a negative exponent not in fraction

$$\int \frac{7}{9\left(\frac{7}{5}\right)^3} \; u^{-4} \; du$$

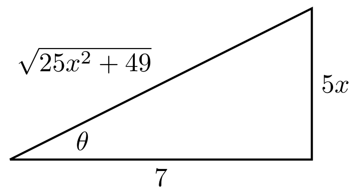
We finally integrate

$$\frac{7}{9\left(\frac{7}{5}\right)^3} \; \frac{1}{-3} u^{-3} + C$$

Replace u with sine

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}(\sin(\theta))^{-3}+C$$

Now we use the triangle from earlier



For our triangle,

$$\sin(\theta)=\frac{\text{OPP}}{\text{HYP}}=\frac{5x}{\sqrt{25x^2+49}}$$

So we replace our $\sin(\theta)$ with that fraction

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}(\sin(\theta))^{-3}+C$$

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}\left(\frac{5x}{\sqrt{25x^2+49}}\right)^{-3}+C$$

We can use up the negative on the exponent by taking the reciprocal.

$$\frac{7}{9\left(\frac{7}{5}\right)^3}\ \frac{1}{-3}\left(\frac{\sqrt{25x^2+49}}{5x}\right)^{3}+C$$

Remember: Square roots can be rewritten as 1/2 powers, and
dividing by a fraction is the same as multiplying by its reciprocal.

$$\frac{7*5^3}{9(7)^3}\ \frac{1}{-3}\left(\frac{(25x^2+49)^{1/2}}{5x}\right)^{3}+C$$

Distribute the 3 power to the fraction.

$$\frac{7*5^3}{9(7)^3}\ \frac{1}{-3}\left(\frac{(25x^2+49)^{3/2}}{5^3x^3}\right)+C$$

Cancel a 7 and a 5^3 on both top and bottom of fraction

$$\frac{1}{9(7)^2}\ \frac{1}{-3}\left(\frac{(25x^2+49)^{3/2}}{x^3}\right)+C$$

Multiply

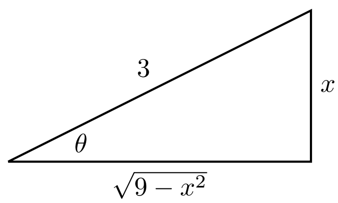
$$-\frac{(25x^2+49)^{3/2}}{1323x^3}+C$$

That is our final answer

Example Problem #2 - Fast Work

$$\int \sqrt{9-x^2}\ dx$$

let $x = 3\sin(\theta)$
then $du = 3\cos(\theta)\ d\theta$



$$\int \sqrt{9-x^2}\ dx$$

$$\int \sqrt{9-(3\sin(\theta))^2}\quad 3\cos(\theta)\ d\theta$$

$$\int \sqrt{9-9\sin^2(\theta)}\quad 3\cos(\theta)\ d\theta$$

$$\int \sqrt{9(1-\sin^2(\theta))}\quad 3\cos(\theta)\ d\theta$$

$$\int \sqrt{9\cos^2(\theta)}\quad 3\cos(\theta)\ d\theta$$

$$\int 3|\cos(\theta)| * 3\cos(\theta)\ d\theta$$

Assume $\theta \in$ Q1

$$\int 3\cos(\theta) * 3\cos(\theta)\ d\theta$$

$$\int 9\cos^2(\theta)\ d\theta$$

Here we use cosine's power-reducing formula
A large list of useful Trig formulas is avaiable at
www.ReetutorsMath.org/notes

$$\int 9\left(\frac{1}{2}\right)(1+cos(2\theta))\ d\theta$$

$$\int \left(\frac{9}{2} + \frac{9}{2}cos(2\theta)\right)\ d\theta$$

let $u = 2\theta$
then $du = 2\ d\theta$

$$\int \left(\frac{9}{2} + \frac{9}{2}cos(2\theta)\right) * \frac{1}{2} * 2\ d\theta$$

$$\int \left(\frac{9}{2} + \frac{9}{2}cos(u)\right) * \frac{1}{2}\ du$$

$$\int \left(\frac{9}{4} + \frac{9}{4}cos(u)\right)\ du$$

$$\frac{9}{4}u + \frac{9}{4}\sin(u) + C$$

$$\frac{9}{4}2\theta + \frac{9}{4}\sin(2\theta) + C$$

$$\frac{9}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\frac{x}{3}\frac{\sqrt{9-x^2}}{3} + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9-x^2}}{2} + C$$

Example Problem #2 - Slow Work

$$\int \sqrt{9-x^2}\ dx$$

This problem has a negative squared variable and a positive constant inside a square root, which indicates we should consider a sine substution. Our a will be $\sqrt{9} = 3$. Because the x^2 has a coefficient of 1, our b will be $\sqrt{1} = 1$. Because $\frac{3}{1} = 3$, we will use the substitution...

let $x = 3\sin(\theta)$
then $du = 3\cos(\theta)\ d\theta$

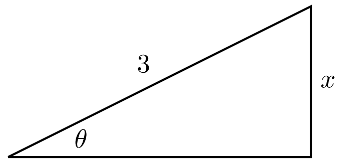
I recommend drawing the trig subsitution's triangle as soon as we choose the trig subsitution. We begin by taking our subsitution and solving for the trig function.

$x = 3\sin(\theta)$
$\frac{x}{3} = \sin(\theta)$

Then we use the definition of sine

$$\sin(\theta) = \frac{x}{3} = \frac{\text{Opposite}}{\text{Hypotenouse}}$$

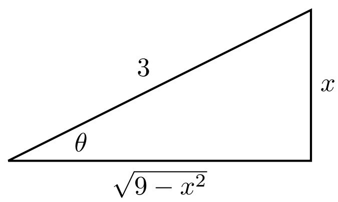
Now we draw a right triangle, and label it with an opposite side of x and a hypotenouse of 3.



The trig substution should always give us 2 sides of the triangle, and we find the third side using the Pythagorean Formula.

$$a^2 + b^2 = c^2$$
$$a^2 + x^2 = 3^2$$
$$a^2 = 3^2 - x^2$$
$$a = \sqrt{9 - x^2}$$

So our finished triangle looks like this:



This will be needed when undoing the trig substution after we integrate. Now back to the integral.

$$\int \sqrt{9 - x^2}\ dx$$

let $x = 3\sin(\theta)$
then $du = 3\cos(\theta)\ d\theta$

$$\int \sqrt{9 - x^2}\ dx$$

$$\int \sqrt{9 - (3\sin(\theta))^2} \quad 3\cos(\theta)\ d\theta$$

$$\int \sqrt{9 - 9\sin^2(\theta)} \quad 3\cos(\theta)\ d\theta$$

Factor out the 9

$$\int \sqrt{9(1 - \sin^2(\theta))} \quad 3\cos(\theta)\ d\theta$$

Pythagorean Identity

$$\int \sqrt{9\cos^2(\theta)} \quad 3\cos(\theta)\ d\theta$$

$$\int 3|\cos(\theta)| * 3\cos(\theta)\ d\theta$$

Technically, the square root of square of an expression is the absolute value of that expression.
But for indefinite integrals, it's generally acceptable to just say

Assume $\theta \in$ Q1

And because all trig functions are positive in Quadrant 1, we remove the absolute value bars.

$$\int 3\cos(\theta) * 3\cos(\theta)\ d\theta$$

$$\int 9\cos^2(\theta)\ d\theta$$

Here we use cosine's power-reducing formula.
A large list of useful Trig formulas is avaiable at
www.ReetutorsMath.org/notes

$$\int 9\left(\frac{1+cos(2\theta)}{2}\right)\ d\theta$$

$$\int 9\left(\frac{1}{2}\right)(1+cos(2\theta))\ d\theta$$

$$\int \left(\frac{9}{2}+\frac{9}{2}cos(2\theta)\right)\ d\theta$$

Because we have 2θ and not just θ in the cosine, we need to use a substution.

let $u = 2\theta$
then $du = 2\ d\theta$

Because we do not have the 2 in $2d\theta$, we need to put it there.
So we multiply by 2. In order to not change our answer, we must also multiply by $\frac{1}{2}$.

$$\int \left(\frac{9}{2}+\frac{9}{2}cos(2\theta)\right) * \left[\frac{1}{2} * 2\right]\ d\theta$$

Now that we have $2d\theta$, we can change it into the du it equals. We also replace 2θ with u.

$$\int \left(\frac{9}{2}+\frac{9}{2}cos(u)\right) * \frac{1}{2}\ du$$

$$\int \left(\frac{9}{4}+\frac{9}{4}cos(u)\right)\ du$$

The integral of a cosine is a sine, and the integral of a constant 9/4 is just that constant times the variable.

$$\frac{9}{4}u+\frac{9}{4}\sin(u)+C$$

Remember $u = 2\theta$

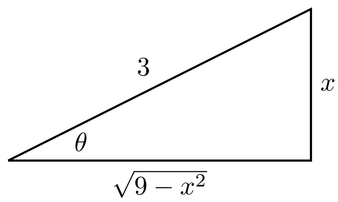
$$\frac{9}{4}2\theta + \frac{9}{4}\sin(2\theta) + C$$

Use the sine double angle formula.

Again, a list of the trig formulas is avaialable at www.ReeTutorsMath.org/notes

$$\frac{9}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) + C$$

Next we undo the trigonometric substitution. This is where we need the triangle from earlier.



For this triangle, the opposide side has length x,
the adjacent side has length $\sqrt{9 - x^2}$, and the hypotenouse has length 3. Therefore...

$$\sin(\theta) = \frac{\text{OPP}}{\text{HYP}} = \frac{x}{3}$$

$$\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}} = \frac{\sqrt{9 - x^2}}{3}$$

And if $\sin(\theta) = (\frac{x}{3})$ then $\theta = \sin^{-1}(\frac{x}{3})$.

Once we know all that, we can change our θ expressions to x expressions.

$$\frac{9}{2}\theta + \frac{9}{2}\left(\sin(\theta)\right)\left(\cos(\theta)\right) + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9 - x^2}}{3}\right) + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{9x\sqrt{9 - x^2}}{18} + C$$

$$\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) + \frac{x\sqrt{9 - x^2}}{2} + C$$

And that is our final answer.